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# ANALYTICAL CONSTRUCTION OF VISCOUS GAS FLOWS USING THE SEQUENCE OF LINEARIZED NAVIER - STOKES SYSTEMS*

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Solutions of the complete Navier-Stokes system are constructed in the form of special series for a viscous, heat conducting continuous compressible medium. The zeroth-order term of the series transmits some exact solution of the initial system (e.g. all parameters of the medium are constants). Further terms of the series are determined by recurrence methods in the course of solving the linearized Navier-Stokes system, homogeneous for the first term and inhomogeneous for all remaining terms. The representations obtained are used to obtain approximate solutions of some boundary value problems. The process of stabilizing **unidirectional flow between** two fixed walls with constant heat flux specified on them is discussed, and an analogue of Poiseuille flow is constructed.

1. We consider the system of Navier-Stokes equations /1/

$$
\begin{gathered}
\frac{\partial\rho}{\partial t}+\mathbf{V}\cdot\nabla\rho+\rho\operatorname{div}\mathbf{V}=0 \qquad (1.1)\\
\rho\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\left\|\frac{\partial v_\alpha}{\partial x_\beta}\right\|^T\right)+\mathrm{Eu}_1c_1^2\nabla\rho+\mathrm{Eu}_2b_1\nabla T=\\
\frac{1}{\mathrm{Re}}\Big[(\operatorname{div}\mathbf{V})\left(\nabla\mu'-\frac{2}{3}\nabla\mu\right)+\nabla\mu\left(\left\|\frac{\partial v_\alpha}{\partial x_\beta}\right\|+\left\|\frac{\partial v_\alpha}{\partial x_\beta}\right\|^T\right)+\\
\left(\mu'+\frac{1}{3}\mu\right)\nabla(\operatorname{div}\mathbf{V})+\mu\Delta\mathbf{V}\Big]\\
\rho c_v\left(\frac{\partial T}{\partial t}+\mathbf{V}\cdot\nabla T\right)+\mathrm{Eu}_2\theta_1b_1T\operatorname{div}\mathbf{V}=\\
\frac{1}{\mathrm{Pr}_1\mathrm{Re}}(\varkappa\Delta T+\nabla\varkappa\cdot\nabla T)+\frac{\theta_1}{\mathrm{Re}}\Big\{\mu'(\operatorname{div}\mathbf{V})^2+\\
\frac{2}{3}\mu\left[\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_2}{\partial x_2}\right)^2+\left(\frac{\partial v_1}{\partial x_1}-\frac{\partial v_3}{\partial x_3}\right)^2+\left(\frac{\partial v_2}{\partial x_2}-\frac{\partial v_3}{\partial x_3}\right)^2\right]+\\
\mu\left[\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right)^2+\left(\frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1}\right)^2+\left(\frac{\partial v_2}{\partial x_3}+\frac{\partial v_3}{\partial x_2}\right)^2\right]\Big\}\\
\mathrm{Eu}_1=\frac{c_1^{*2}}{u_0^2},\quad \mathrm{Eu}_2=\frac{b_1{}^*T_0}{\rho_0u_0^2},\quad \mathrm{Re}=\frac{\rho_0u_0L}{\mu^*}
\end{gathered}
$$

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$$\theta_1 = \frac{u_0^2}{c_v{}^* T_0}, \quad \mathrm{Pr}_1 = \frac{\mu^* c_v{}^*}{\varkappa^*}$$

which represents a differential form of the laws of conservation of mass, momentum and energy for the flows of a viscous, heat-conducting continuous compressible medium. Here $t$ is the time, $x_\alpha$ $(\alpha = 1, 2, 3)$ are the spatial coordinates, $\mathbf{x} = \{x_1, x_2, x_3\}$, $\rho$ is the density, $v_\alpha$ $(\alpha = 1, 2, 3)$ are Cartesian projections of the velocity vector $\mathbf{V}$ of the medium, $T$ is the temperature, $\mu, \mu'$ are the coefficients of dynamic and volume viscosity (the first and second coefficients of viscosity), $\varkappa$ is the thermal conductivity, $\| \partial v_\alpha / \partial x_\beta \|$ is a Jacobi matrix, $\| \partial v_\alpha / \partial x_\beta \|^T$ is its transpose, $\Delta$, div, $\nabla$ denote the Laplace, divergence and grad operators, a dot denotes a scalar product, the vectors are regarded as line vectors and the product of a vector and a matrix is calculated by the usual rules of matrix multiplication.

When deriving system (1.1) it was assumed that there are no external mass forces nor external heat sources or sinks, and $\rho, T$ were chosen as the independent thermodynamic parameters. The equations of state as well as the coefficients of viscosity and thermal conductivity are assumed to be given functions

$$\mu = \mu(\rho, T), \quad \mu' = \mu'(\rho, T), \quad \varkappa = \varkappa(\rho, T) \tag{1.2}$$

$$p = p(\rho, T), \quad e = e(\rho, T) \tag{1.3}$$

where $p$ denotes the pressure and $e$ is internal energy. Then

$$c_1^2 = \frac{\partial p}{\partial \rho}, \quad b_1 = \frac{\partial p}{\partial T}, \quad c_v = \frac{\partial e}{\partial T}, \quad p - \rho^2 \frac{\partial e}{\partial \rho} = b_1 T$$

The passage to dimensionless coordinates is made in the system (1.1) in a standard manner, using the positive constants $L, \rho_0, u_0, T_0$, and an asterisk denotes the dimensional values of the functions for $\rho = \rho_0$, $T = T_0$. Since we shall construct the solutions of the system (1.1) in the form of series, it follows that the functions (1.2) and (1.3) are assumed to be analytic in the neighbourhood of the point $(\rho = 1, T = 1)$ and positive at the point itself.

In the case of a thermodynamically ideal gas with equations of state

$$p = R\rho T, \quad e = c_{v0} T, \quad R, c_{v0} = \mathrm{const} > 0 \tag{1.4}$$

we have $\mathrm{Eu}_1 = \mathrm{Eu}_2 = 1/(\gamma M^2)$, $M^2 = u_0^2/(RT_0\gamma)$ is the square of the Mach number, $\gamma = 1 + R/c_{v0} > 1$ is the adiabatic index of the gas, $\theta_1 = \gamma(\gamma - 1)M^2$; $\mathrm{Pr}_1 = \mathrm{Pr}/\gamma$, $\mathrm{Pr} = c_{v0}\mu^*\gamma/\varkappa^*$ is the Prandtl number. If, using the given $\rho_0$, $T_0$ we choose $u_0 = [c_1{}^{*2} + b_1{}^{*2} T_0/(\rho_0{}^2 c_v{}^*)]^{1/2}$ as the scale of velocity, then $\mathrm{Eu}_1 + \mathrm{Eu}_2{}^2\theta_1 = 1$. In the case of a thermodynamically ideal gas this corresponds to the fact that the velocity of sound $u_0 = (RT_0\gamma)^{1/2}$, is chosen as the velocity scale, and then we have $M = 1$.

In the present paper the solutions of system (1.1) are constructed in the form of series

$$\mathbf{U} = \sum_{k=0}^{\infty} \mathbf{U}_k(t, \mathbf{x}) \varepsilon^k, \quad \mathbf{U} = \{\rho, v_1, v_2, v_3, T\} \tag{1.5}$$

in powers of the new independent variable $\varepsilon$, which has no predetermined specific physical meaning. We shall represent the vector $\mathbf{U}$ is the form (1.5), if $\mathbf{U}_0$ represents any exact solution of the system (1.1) transmitting, for example, a homogeneous medium at rest.

In order to obtain a system of equations for the components of the vector $\mathbf{U}_k$ $(k \geqslant 1)$, we assume that $\mathbf{U}$ also depends on $\varepsilon$, system (1.1) is differentiable $k$ times in $\varepsilon$, and $\varepsilon$ is assumed equal to zero. As a result we obtain, for the components of the vector $\mathbf{U}_k$, a linear system of partial differential equations with the same principal part for all $k \geqslant 1$, the homogeneous part for $k = 1$ and inhomogeneous for $k \geqslant 2$. The coefficients of the principal part of the system depend on $\mathbf{U}_0$, and the inhomogeneities at $k \geqslant 2$ are polynomials in components of the vectors $\mathbf{U}_l$, $0 \leqslant l \leqslant k - 1$ and of their derivatives in $t$, $x_\alpha$ $(\alpha = 1, 2, 3)$. Specifying the initial and single-type boundary conditions for $\mathbf{U}_k$ generates uniquely for the system (1.1) the conditions additionally dependent on $\varepsilon$, and vice versa. The system for $\mathbf{U}_1$ is identical with the system obtained when the initial system (1.1) is linearized on a given solution $\mathbf{U}_0$ /2/. Therefore the series (1.5) can be regarded as a solution obtained as a result of linearizing the complete Navier-Stokes system with subsequent construction of all higher-order approximations. The convergence of series (1.5) must be established when specific boundary value problems are considered.

The basic aim of the present paper is the effective construction of the coefficients of the series (1.5), so that finite sections of the series can be used for an approximate solution of certain problems. Therefore, the convergence of the series is shown for the simplest situation, when the proof can be reduced to the analogue of the Cauchy-Kovalevskaya theorem.

Let the initial terms of the series $\mathbf{U}_k$ be constructed for $0 \leqslant k \leqslant k_0$ as the solutions of the corresponding systems in the form of functions analytic in the neighbourhood of the point $(t = 0, \mathbf{x} = \mathbf{x}^\circ)$. Let them also satisfy the initial and boundary conditions traditionally used for the Navier-Stokes system /3/. The remaining coefficients of the series $\mathbf{U}_k$ for $k \geqslant k_0 + 1$ are constructed in the course of solving the Cauchy problem for the corresponding systems of equations with the analytic data $\mathbf{U}_k|_{\varphi=0}$, $\mathbf{Z}_{k\varphi}|_{\varphi=0}$ on the surface $\varphi = 0$. Here we have $\mathbf{Z} = \{\mathbf{V}, T\}$.

We assume that in the case of the surface $\varphi = 0$ the differential systems for $\mathbf{U}_k$ are of the Kovalevskaya type, i.e. $\mathbf{U}_k$ can be determined uniquely in the form of analytic functions, and the series in powers of $\varepsilon$ formed from the given values of $\mathbf{U}_k|_{\varphi=0}$, $\mathbf{Z}_{k\varphi}|_{\varphi=0}$, converge when $|\varepsilon| < \varepsilon_0$, $\varepsilon_0 > 0$. The last assumption will hold if, for example, the data on the surface $\varphi = 0$ are taken, for $\mathbf{U}_k$, $k \geqslant k_0 + 1$, as their zero values. This is equivalent to taking the finite sums

$$\mathbf{U}|_{\varphi=0} = \Big[\sum_{k=0}^{k_0} \mathbf{U}_k(t, \mathbf{x})\,\varepsilon^k\Big]\Big|_{\varphi=0}$$

$$\mathbf{Z}_\varphi|_{\varphi=0} = \Big[\frac{\partial}{\partial\varphi} \sum_{k=0}^{k_0} \mathbf{Z}_k(t, \mathbf{x})\,\varepsilon^k\Big]\Big|_{\varphi=0}$$

as the data on the surface $\varphi = 0$ for $\mathbf{U}$ and $\mathbf{Z}$.

The surface $\varphi = 0$ can be taken in the form $\varphi = x_1 - x_1(t, x_2, x_3)$ under the condition that $x_1^\circ = x_1(0, x_2^\circ, x_3^\circ)$ and $\varphi = 0$, with the values of $\mathbf{U}, \mathbf{Z}_\varphi$ specified on it, is not a contact surface.

When the above conditions hold, the series (1.5) will converge in some neighbourhood of the point $(t = 0, \mathbf{x} = \mathbf{x}^\circ, \varepsilon = 0)$.

The above assertion is not proved here, since it is a special case of the theorem proved in /4/: since the system (1.1) has no derivatives $\partial\mathbf{U}/\partial\varepsilon$, it follows that the surface $\varepsilon = 0$ is formally a characteristic. The condition that $\mathbf{U}_0$ will represent any exact solution of the system (1.1), is identical with the necessary condition of solvability of the corresponding characteristic Cauchy problem. The surface $\varphi = 0$ represents the surface on which additional conditions are specified, ensuring the uniqueness of the solution of the characteristic Cauchy problem. The theorem given in /4/ ensures the local convergence of the series (1.5). The domain of convergence in the $(t, \mathbf{x})$ space increases as $|\varepsilon|$ decreases and "reaches" the points at which the functions $\mathbf{U}_k$ $(0 \leqslant k \leqslant k_0)$, $\varphi$ and components of the matrix $S^{-1}|_{\varphi=0}$ have singularities. Here $S = S(t, \mathbf{x})|_{\varphi=0}$ is a matrix preceding the derivatives of $\mathbf{U}_k$ of higher order in $\varphi$ in the corresponding differential systems.

The quantity $\varepsilon$ represents the deviation of the solution $\mathbf{U}$ from $\mathbf{U}_0$, although $\mathbf{U}$ may satisfy different boundary conditions as compared with $\mathbf{U}_0$. The assertion formulated here establishes the connection between the linearizing procedure with subsequent construction of the higher-order approximations, and the process of constructing the solutions of hyperbolic systems in the form of characteristic series /4, 5/. When $\mathbf{U}_k$ $(k \geqslant k_0 + 1)$ is constructed using the method given above the series (1.5) will transmit exactly the local solution of the Cauchy problem with the data on $\varphi = 0$, and transmit in an approximate manner the solution of the initial boundary value problem traditionally formulated for the Navier-Stokes system. It is possible that, using more refined estimates and taking into account the specific initial and boundary conditions (as was done in e.g. /3, 6/), will give the domain of applicability of the representation (1.5) with increased accuracy. We shall, however, stress once again, that the main aim of this paper is to show that in certain situations we can constructively determine the coefficients of the series assigning exact solutions of system (1.1). Finite segments of these series are used to obtain approximate solutions of certain initial-boundary value problems.

2. Next we take the solution $\mathbf{U}_0 = \{1, 0, 0, 0, 1\}$ which transmits a homogeneous medium at rest. Then, for any analytic function (1.2) and (1.3) we obtain, for $\mathbf{U}_k$ $(k \geqslant 1)$, the linear systems with constant coefficients

$$\begin{aligned}
&\frac{\partial\rho_k}{\partial t} + \operatorname{div}\mathbf{V}_k = F_k \\
&\frac{\partial\mathbf{V}_k}{\partial t} + \mathrm{Eu}_1\nabla\rho_k + \mathrm{Eu}_2\nabla T_k - \\
&\quad \frac{1}{\mathrm{Re}}\Big\{\Big[\mu'(1, 1) + \frac{1}{3}\Big]\nabla(\operatorname{div}\mathbf{V}_k) + \Delta\mathbf{V}_k\Big\} = \mathbf{G}_k \\
&\frac{\partial T_k}{\partial t} + \mathrm{Eu}_2\theta_1\operatorname{div}\mathbf{V}_k - \frac{1}{\mathrm{Pr}_1\mathrm{Re}}\Delta T_k = H_k
\end{aligned} \tag{2.1}$$

When $k=1$, the quantities $F_k, \mathbf{G}_k, H_k$ are zeros. When $k \geqslant 2$,

$$F_k = -\sum_{l=1}^{k-1} [V_l \cdot \nabla \rho_{k-l} + \rho_l \operatorname{div} V_{k-l}]$$

The quantities $\mathbf{G}_k, H_k$ are also polynomials in terms of the components of the vectors $\mathbf{U}_l$ $0 \leqslant l \leqslant k-1$ and their derivatives (the actual expressions for $\mathbf{G}_k, H_k$ are quite bulky and are therefore not given here).

System (2.1) can be transformed as follows. We introduce new unknown functions $P = \mathrm{Eu}_1\rho_k + \mathrm{Eu}_2 T_k$, $W = c_0 \operatorname{div} \mathbf{V}_k$, $c_0 = (\mathrm{Eu}_1 + \mathrm{Eu}_2{}^2\theta_1)^{1/2}$, change the time scale $t' = c_0 t$ (we shall omit the primes from now on), differentiate each equation of motion of system (2.1) with respect to the corresponding $x_\alpha$, put together the resulting expressions, differentiate with respect to $t$ the first and last equation of (2.1), and take their linear combination. As a result we obtain the system

$$\begin{gathered} W_t = \mu_0 \Delta W - \Delta P + g_k, \quad P_{tt} = \Delta P + \varkappa_0 \Delta P_t + \\ (\varkappa_0 a - \mu_0)\, \Delta W + h_k \\ \mu_0 = \frac{(\mu' + 4/3)}{c_0 \mathrm{Re}}, \quad \varkappa_0 = \frac{1}{c_0 \mathrm{Re} \mathrm{Pr}_1}, \quad a = \frac{\mathrm{Eu}_1}{c_0{}^2}, \quad 0 \leqslant a \leqslant 1 \end{gathered} \tag{2.2}$$

where the terms $g_k$ and $h_k$ can be determined uniquely from the right-hand sides of system (2.1), and the constant $a$ represents the compressibility of the medium. In the case of an ideal gas, $c_0 = 1/M$, $a = 1/\gamma$. The initial conditions at $t=0$ for $\mathbf{U}_k$ generate uniquely the initial conditions for $W, P$. If the solution of system (2.2) is known, then $\rho_k$ can be found by integrating the known expression with respect to t and $\mathbf{V}_k$, $T_k$ is found from the corresponding linear equations of heat conduction. In the special case of $\mu_0 = a\varkappa_0$ (this is equivalent for the equations of state (1.4) to the fact that $\mathrm{Pr} = 0.75$), the second equation of the system will have the form

$$P_{tt} = \Delta P + \varkappa_0 \Delta P_t + h_k \tag{2.3}$$

After solving this equation we find $W$ from the inhomogeneous equation of heat conduction. In the case when $\mathrm{Eu}_2 = 0$, the first four equations of system (2.1) will also yield an equation of the form (2.3) for $W$.

The homogeneous system (2.2) can be called the "linear system of viscous flows", and its special case, i.e. the homogeneous Eq.(2.3), the "linear equation of viscous flows", since they describe, in particular, the propagation of small perturbations through a homogeneous, viscous heat-conducting compressible continuous medium.

The general solution of the Cauchy problem and some boundary value problems are given in integral form in /7/ for the homogeneous Eq.(2.3), and the properties of this solution are studied for large $t$ and small $\varkappa_0$. Below we utilize the fact that the homogeneous system (2.2) allows the separation of variables $W = W(t)\,X(\mathbf{x})$, $P = P(t)\,X(\mathbf{x})$, $\Delta X = -n^2 X$, where $n$ is a positive integer. In particular, if we take the harmonics of a single spatial variable as $X$, then $n$ will be the frequency of the harmonic while $W(t)$ and $P(t)$ will be the amplitudes. The roots of the characteristic equation of the system of ordinary differential equations for $W(t)$, $P(t)$ will be the roots of the equation

$$\nu^3 + n^2(\varkappa_0 + \mu_0)\,\nu^2 + n^2(1 + n^2\varkappa_0\mu_0)\,\nu + n^4\varkappa_0 a = 0 \tag{2.4}$$

When $\mu_0 = a\varkappa_0$, we have $\nu_{n1} = -\mu_0 n^2$ and

$$\nu_{n2,3} = -n^2(\varkappa_0/2 \pm \sqrt{\varkappa_0{}^2/4 - 1/n^2}) \tag{2.5}$$

When $a=0$ we have $\nu_{n1} = 0$, and $\nu_{n2,3}$ will be given by formula (2.5) in which $\varkappa_0$ in the first term must be replaced by $(\varkappa_0 + \mu_0)$, and in the expression under the square root sign by $(\varkappa_0 - \mu_0)$. When $a = 1$ $\nu_{n1} = -n^2\varkappa_0$, $\nu_{n2,3}$ are given by formula (2.5) in which $\varkappa_0$ must be replaced by $\mu_0$. When $0 < a < 1$, the roots of Eq.(2.4) will contain no purely imaginary numbers: $-n^2(\varkappa_0 + \mu_0) < \nu_{n1} < 0$ if $\nu_{n2,3}$ are real then $\nu_{n1} < \nu_{n2,3} < 0$, while if $\nu_{n2,3}$ are complex, then their real parts are negative.

In the general case the values of the roots can be written using the formulas

$$\begin{gathered} \nu_{n1} = A_+ + A_- - A_0, \ \nu_{n2,3} = -(A_+ + A_-)/2 + A_0 \pm \\ (A_+ - A_-)\ \sqrt{3}i/2, \ i = \sqrt{-1} \\ A_\pm = (-q_2/2 \pm \sqrt{Q})^{1/3}, \ A_+A_- = -q_1/3, \ A_0 = n^2(\varkappa_0 + \mu_0)/3 \\ Q = (q_1/3)^3 + (q_2/2)^2, \ q_1 = -n^4(\varkappa_0 + \mu_0)^2/3 + n^2(1 + n^2\varkappa_0\mu_0) \\ q^2 = 2n^6(\varkappa_0 + \mu_0)^3/27 - n^4(\varkappa_0 + \mu_0)(1 + n^2\varkappa_0\mu_0)/3 + n^4\varkappa_0 a \end{gathered}$$

If $Q \leqslant 0$, then $\nu_{n2,3}$ are real and different when $Q < 0$, while when $Q = 0$ $\nu_{n2} = \nu_{n3}$. If $Q > 0$, then $\nu_{n2,3}$ are complex conjugate. Analysing the expressions for $Q$ we find that $\varkappa_0 \neq \mu_0$, beginning with some $n$, $Q < 0$. If $\varkappa_0 = \mu_0$, i.e. $\mathrm{Pr}_1 = 1/(\mu' + 4/3)$, we have when

$0 < a < {}^8/_9, Q > 0$ for all $n$. When $a = {}^8/_9\, Q = 0$ for $n = \sqrt{3}/\varkappa_0$ and $Q > 0$ for all other $n$. When ${}^8/_9 < a < 1$ $Q < 0$ for $n$ satisfying the inequalities $\sqrt{d_-}/\varkappa_0 < n < \sqrt{d_+}/\varkappa_0$, $Q = 0$ for $n = \sqrt{d_\pm}/\varkappa_0$ and $Q > 0$ for all other $n$. Here we have

$$d_\pm = \Big\{\Big[1 - \frac{27}{4}\Big(a - \frac{2}{3}\Big)^2\Big] \pm \Big(\Big[1 - \frac{27}{4}\Big(a - \frac{2}{3}\Big)^2\Big]^2 - 4(1-a)\Big)^{1/2}\Big\}[2(1-a)]^{-1} > 0$$

The values of $\mathbf{U}_1$ obtained by separating the variables can be used, as particular solutions of the homogeneous system (2.2), to study the development of small perturbations of varying size in homogeneous flows of viscous media for various values of $\mu_0, \varkappa_0, a$. In particular, when $\mu_0 = a\varkappa_0$, then from (2.5) it follows that for small values of $\varkappa_0 (0 < \varkappa_0 < 2)$ the amplitude of the first harmonic $(1 \leqslant n < 2/\varkappa_0)$ will oscillate while decreasing, and we will obtain, with help of the linear combination of low-frequency harmonics, a decaying running wave as a solution of the homogeneous system (2.2). The velocity of wave propagation is equal to $\sqrt{1 - n^2\varkappa_0^2/4}$, the wave amplitude is large $(t < 2/(\varkappa_0 n^2))$ and does not decay. For large values of $\varkappa_0$ the high-frequency harmonics are stationary and do not decay $(|P(t)| > |P(0)|/3)$ for long periods $(t < \varkappa_0)$. When $\varkappa_0 = \mu_0$, all high-frequency harmonics beginning from a specified value oscillate while decaying, and can be used to construct the corresponding decaying running waves as solutions of the homogeneous system (2.2).

The possibility of separating the variables when the system is homogeneous, makes the efficient construction of the coefficients of series (1.5) feasible.

We shall illustrate this by considering one-dimensional $(\partial/\partial x_2 = \partial/\partial x_3 = v_2 = v_3 = 0)$ non-steady flows of an ideal gas between two impermeable walls $x_1 = 0$ and $x_1 = \pi$ acted upon by a given heat flux

$$\partial T/\partial x_1 |_{x_1=0,\ x_1=\pi} = A\varepsilon, \quad A = \text{const}$$

with the following constant values:

$$\mu' = 0, \quad \mu = \mu_0 = 4/(3\,\mathrm{Re}), \quad \varkappa = \varkappa_0 = \gamma/(\mathrm{Pr\,Re}) \tag{2.6}$$

In the case of $A = 0$, a finite segment of the series (1.5) describes approximately the process of stabilizing a flow, in which the distributions of gas-dynamic parameters are given for the instant $t = 0$, towards a state of uniform rest as $t \to +\infty$. In this case $\varepsilon$ determines the difference between the initial and the limit flow.

When $A \neq 0$, we study the passage of the gas from a state of uniform rest at $t = 0$, to a state at rest with constant temperature gradient $\partial T/\partial x_1 = A\varepsilon$. The passage is caused by a constant heat flux applied to the walls $x_1 = 0$ and $x_1 = \pi$ at $t \geqslant 0$. In this case the physical meaning and the value of $\varepsilon$ is actually determined by the intensity of this heat flux, i.e. by the value of the constant $A\varepsilon$.

In order to construct an approximate solution of such a problem, we take as $\mathbf{U}_1$ the sum of solutions of the homogeneous system (2.1):

$$\rho_1 = -T_1, \quad v_{11} = 0, \quad T_1 = A\varepsilon x_1$$

and the non-stationary solution of the form

$$f_1(t, x_1) = \sum_{n=0}^{N} f_{1n}(t) \cos nx_1, \quad f - \rho, T \tag{2.7}$$

$$v_{11}(t, x_1) = \sum_{n=0}^{N} v_{11n}(t) \sin nx_1$$

This will yield, for the functions $f_{1n}(t), v_{11n}(t)$, a system of linear ordinary differential equations with constant coefficients. The initial conditions for this system are chosen such, that when $t = 0$, the sum of the stationary and non-stationary solutions will transmit, approximately, $\mathbf{U}_1 = 0$: $\rho_{10}(0) = v_{11n}(0) = T_{10}(0) = 0$; $\rho_{1n}(0), T_{1n}(0), 1 \leqslant n \leqslant N$ are the corresponding Fourier coefficients when the functions $\pm A\varepsilon x_1$ are expanded in a series in $\cos nx_1$.

If, after constructing the solution, we replace approximately $f_1$ and the stationary part of the functions $\rho_1, T_1$ by corresponding finite segments of the Fourier series for $n \leqslant N$, the right-hand sides of system (2.1) for $\mathbf{U}_2$ will be finite trigonometric sums in $\cos nx_1$ for $F_2$ and $H_2$ and in $\sin nx_1$ for $G_2$, $n \leqslant 2N$. The coefficients of these sums are known functions of $t$. Therefore $f_2$ and $v_{12}$ can also be represented in the form (2.7) with their coefficients $f_{2n}(t)$, $v_{12n}(t), n \leqslant 2N$. The functions are found from the inhomogeneous systems of ordinary differential equations with zero initial conditions.

It can be shown that in this case we also have the representation (2.7) for all subsequent coefficients of $U_k$, for $n \leqslant kN$, and the functions $f_{kn}(t)$ and $v_{1kn}(t)$ can be determined uniquely as solutions of the corresponding differential systems with zero initial conditions. The value of $\mathbf{U}_1$ and representation (2.7) at $k \geqslant 2$ ensure for the solution (1.5) that conditions of adhesion and constancy of heat flux hold at the walls $x_1 = 0$ and $x_1 = \pi$. The solution (1.5) transmits approximately, at the instant $t = 0$, the state of uniform rest.

Figs.1-3 show the results of calculating the parameters of gas flow in the case $\mu_0 = 10^{-3}$; $\varkappa_0 = 0.05$; $\gamma = 1.4$; $A = 1$; $\varepsilon = 0.1$. In the versions calculated we have $k_0 = 5$, $N = 6$, i.e. we have taken into account the terms of the series (1.5) up to and including $\varepsilon^5$, and the number of harmonics taken in $\mathbf{U}_k$ was $6k$. At $t \approx 200$ the solution reaches the limiting stationary state with $v_1 = 0$.

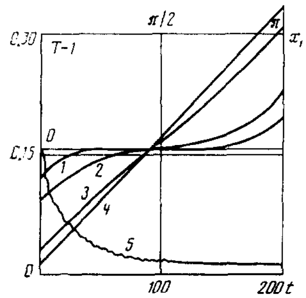


Fig.1

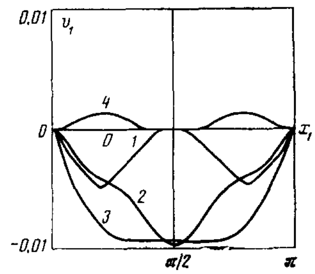


Fig.2

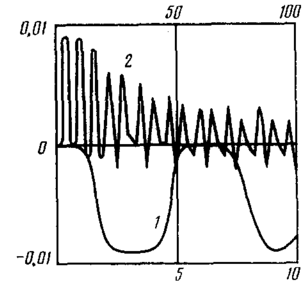


Fig.3

Fig.1 gives the values of the temperature. The lines $0 - 4$ correspond to instants of time $t = 0; 2.5; 10; 50; 200$. $T - 1$ is plotted along the ordinate and $x_1$ along the abscissa with $0 \leqslant x_1 \leqslant \pi$ for curves $0 - 4$, and $t$ when $0 \leqslant t \leqslant 200$ for curve 5 transmitting the value $(T - 1)|_{x_1=0}$. The passage from the state of uniform rest to the state at rest with a constant non-zero value of the temperature gradient is connected with the redistribution of the density on the segment $0 \leqslant x_1 \leqslant \pi$, and is therefore accompanied by a flow of gas. The flow is oscillatory. Fig.2 can give some insight into the initial stage of this flow, as it shows the values of $v_1$ at the instants $t = 0, 1, 2, 3, 6$ (curves $0 - 4$).

Fig.3 shows the dependence of $u = v_1|_{x_1=\pi/2}$ on time. In the case of curve *1*, $t$ was plotted along the abscissa from 0 to 10, and $u$ was plotted along the ordinate. In the case of curve *2*, $-u$ was plotted along the ordinate and $t$ from 0 to 100 along the abscissa. The oscillatory nature of the process of establishment reflects, in particular, the manner in which the values of the temperature was established on the walls (see curve 5 of Fig.1). However, if at some instants of time the values of $|v_1|$ are relatively large, then, as the calculations show, the values of $v_1$ at these instances will have the same sign for all $0 \leqslant x_1 \leqslant \pi$. Therefore the monotonic form of the dependence of $T$ on $x_1$ will be preserved at all times. When the values of $\mu_0, \varkappa_0$ increase and those of $A\varepsilon$ decrease, the dynamics of the process of establishment will manifest themselves less strongly; $|v_1|$ will become smaller, and $v_1 \leqslant 0$ at all $t \geqslant 0$.

In the above example the neglected terms of the series were not estimated. The error of the solution obtained was estimated approximately from the manner in which the terms of the series constructed were decreasing, i.e. from the way in which the computed $\mathbf{U}_k$ behaved as $k$ increased. The following estimate was obtained in the course of computations for the coefficients of the harmonics in $\mathbf{U}_k$:

$$\sum_{n=0}^{kN} |f_{kn}(t)| \leqslant M_k(t)\, A^k, \quad \sum_{n=1}^{kN} |v_{1kn}(t)| \leqslant M_k(t)\, A^k$$

where $M_k(t)$ is a slowly varying function and $M_k(t) \leqslant 2$. Thus the moduli of the computed terms of the series (1.5) decreased in this case not more slowly than the corresponding terms of the geometrical progression $\sum_{k=0}^{\infty} 2\,(A\varepsilon)^k$. When the values of $\mu_0, \varkappa_0$ decreased and those of $A\varepsilon$ increased, the convergence of the series for the case in question deteriorated. Terms of the series (1.5) began to increase rapidly with time, and subsequent terms became larger than the preceding ones.

3. Let us now consider the stationary flows. When the gas is ideal and the functions (1.2) is constant, system (1.1) has exact solutions: a solution with a linear velocity profile /8/; a solution for special values of the index $\gamma$

$$\rho_1 = a_1,\ v_1 = v_2 = 0,\ T = a_2 x_1 + a_3,\ a_j = \text{const},\ 1 \leqslant j \leqslant 3$$

$$v_1 = a_2 \operatorname{Re} x_2^2/(3M^2) \quad \text{when} \quad \gamma = 3/2$$

$$v_1 = a_2 \operatorname{Re} (x_2^2 + x_3^2)/(8M^2) \quad \text{when} \quad \gamma = 2$$

However, the "layered" flows listed above do not allow us, unlike Poiseuille flow in the case of an incompressible medium /1/, to satisfy the conditions of adhesion at two fixed walls when $\mathbf{x} = \{x_1, x_2\}$, or on the walls of a tube when $\mathbf{x} = \{x_1, x_2, x_3\}$. The representation (1.5) enables us to construct, in the case of a compressible medium with arbitrary functions (1.2), (1.3), an analogue of Poiseuille flow, since the homogeneous system (2.1) has the following solution:

$$\rho_1 = 0,\ \mathbf{V}_1 = \{v_{11}(x_1, x_2), 0, 0\},\ T_1 = Ax_1$$
$$\Delta v_{11} = A \operatorname{Re} \operatorname{Eu}_2$$

We can take, as a solution of the last equation, the Poiseuille parabola, $v_{11} = A \operatorname{Re} \operatorname{Eu}_2 x_2 (x_2 - h_0)/2$ ($h_0 = \text{const}$) for the two-dimensional case and $v_{11} = A \operatorname{Re} \operatorname{Eu}_2 (x_2^2 + x_3^2 - r_0^2)/4$ ($r_0 =$ const is the tube radius for the three-dimensional case, as well as other functions which are identical with the velocity distributions in a viscous incompressible medium in tubes of elliptic, rectilinear and triangular cross-section /1/. After determining $\mathbf{U}_1$, further coefficients of the series (1.5) can be found from their inhomogeneous systems (2.1). At the same time, $\mathbf{U}_k$ exhibts a functional arbitrariness which makes it possible to satisfy the conditions of adhesion at prescribed surfaces for $\mathbf{V}_k$ and secure a specified temperature or heat flux for $T_k$.

$\mathbf{U}_k$ were constructed directly in the two-dimensional case $(\partial/\partial x_3 = v_3 = 0)$ for an ideal gas with equations of state (1.4), and constant values of (2.6). The velocity of sound was taken as the velocity scale (i.e. $M = 1$), and

$$\rho_1 = 0,\ v_{11} = 2Ax_2(x_2 - h_0)/(3\gamma\mu_0),\ v_{21} = 0,\ T_1 = Ax_1 \tag{3.1}$$

as $\mathbf{U}_1$.

Then we find that in the system of equations for $\mathbf{U}_2$ $F_2$ and $G_2$ are zeros and $H_2 = 2\mu_0 v_{11}' - Av_{11}$. We take as the solution of this system

$$\rho_2 = \rho_{20} - T_2(x_2),\ v_{12} = v_{22} = 0$$
$$T_2 = T_2(x_2) = -\frac{A^2}{12\gamma\mu_0\varkappa_0}\left[\frac{4}{3}\left(\gamma - \frac{3}{2}\right)\left(x_2 - \frac{h_0}{2}\right)^4 + x_2^2\right] + T_{21}x_2 + T_{20}$$

The constants $T_{20}, T_{21}$ can be chosen from the conditions $T_2|_{x_2=0} = 0$, $\partial T_2/\partial x_2|_{x_2=0} = 0$, as wells as from the conditions $T_2 = 0$ when $x_2 = 0$ and $x_2 = h_0$. In the latter case we have

$$T_2(x_2) = -\frac{A^2}{12\gamma\mu_0\varkappa_0} x_2(x_2 - h_0)\left\{\frac{4}{3}\left(\gamma - \frac{3}{2}\right)\left[\left(x_2 - \frac{h_0}{2}\right)^2 + \frac{h_0^2}{4}\right] + 1\right\}$$

The constant $\rho_{20}$ is found later when the zero solution is satisfied for $v_{24}$ when $x_2 = h_0$. In the course of constructing the subsequent $\mathbf{U}_k$, the inhomogeneities in the system (2.1) will grow, and in order to reduce the unnecessary bulk we shall only show the sequence of determining the components of the vector $\mathbf{U}_k$ and give the final formation for $k \leqslant 5$.

The fourth equation of system (2.1) will yield $T_k$ in such a manner, that the relations $T_k = 0$ at $x_2 = 0, x_2 = h_0$ will also hold. The first equation of system (2.1) will yield an expression for $\partial v_{2k}/\partial x_2$, which will then be differentiated in $x_2$ and substituted into the third equation in place of $\partial^2 v_{2k}/\partial x_2^2$.

As a result, the first three equations will form a system in which $\partial\rho_k/\partial x_2, \partial^2 v_{1k}/\partial x_2^2, \partial v_{2k}/\partial x_2$ are the principal derivatives in $x_2$. The functional arbitrariness which has appeared in the course of solving this differential system makes it possible to satisfy the conditions $v_{1k} = 0$ for $x_2 = 0$ and $x_2 = h_0$, $v_{2k} = 0$ when $x_2 = 0$. In order to satisfy the condition $v_{2k} = 0$ when $x_2 = h_0$, we make use of the functional arbitrariness in $\rho_{k-2}$, and the arbitrariness in $\rho_k$ is used to satisfy the condition $v_{2,k+2} = 0$ when $x_2 = h_0$. Taking the form of $\mathbf{U}_k$ $(k \leqslant 5)$ into account, we obtain the following representation for solving system (1.1):

$$\begin{aligned}
\rho &= 1 + \rho_2(x_2)\varepsilon^2 - Ax_1\rho_2(x_2)\varepsilon^3 + [Ax_1^2\rho_2(x_2) + \\
&\quad (2A)^{-1}Q_3(x_2) + \gamma\mu_0 Q_1(x_2) - T_2(x_2)\rho_2(x_2) + \rho_{40}]\varepsilon^4 + \\
&\quad [Ax_1\rho_2(x_2)T_2(x_2) - 2A\gamma\mu_0 x_1 Q_1(x_2) - Ax_1\rho_4(x_1, x_2) - \\
&\quad x_1 Q_3(x_2) + \rho_{50}]\varepsilon^5 + \varepsilon^6(\ldots) \\
v_1 &= \frac{2A}{3\gamma\mu_0}x_2(x_2 - h_0)\varepsilon + \frac{\varkappa_0}{\gamma}\left[Q_3(x_2) - Q_3(h_0)\frac{x_2}{h_0}\right]\varepsilon^5 + \varepsilon^6(\ldots) \\
v_2 &= Q_2(x_2)\varepsilon^4 - 2Ax_1 Q_2(x_2)\varepsilon^5 + \varepsilon^6(\ldots) \\
T &= 1 + Ax_1\varepsilon + T_2(x_2)\varepsilon^2 - (2A)^{-1}Q_3(x_2)\varepsilon^4 + \\
&\quad x_1 Q_3(x_2)\varepsilon^5 + \varepsilon^6(\ldots) \\
\rho_{20} &= Ah_0^2[{}^8/_3(\gamma - {}^3/_2)h_0^2 + 7]/(420\gamma\mu_0\varkappa_0),\ Q_1(x_2) = \\
&\quad Av_{11}(x_2)\rho_2(x_2)
\end{aligned} \tag{3.2}$$

$$Q_2(x_2) = \int_0^{x_2} Q_1(x_2)\,dx_2, \quad Q_2(h_0) = 0, \quad Q_3(x_2) = -\frac{2A\gamma}{\varkappa_0}\int_0^{x_2} Q_2(x_2)\,dx_2$$

The solution (3.2) yields an expression for the pressure

$$\begin{aligned} p = \gamma^{-1}\{1 + Ax_1\varepsilon + \rho_{20}\varepsilon^2 + [\rho_2(x_2)\,T_2(x_2) - {}^1/_2A^{-1}Q_3(x_2) - \\ A^2x_1^2\rho_2(x_2) + \rho_4(x_1,\ x_2)]\,\varepsilon^4 + [x_1Q_3(x_2) - \\ Ax_1\rho_2(x_2)\,T_2(x_2) + Ax_1\rho_4(x_1, x_2) + \rho_5(x_1, x_2)]\,\varepsilon^5 + \varepsilon^6(\ldots)\} \end{aligned}$$

The physical meaning of the parameter $\varepsilon$ is given by the quantity $A\varepsilon/\gamma$ which represents the principal part of the value of the derivative $\partial p/\partial x_1$ characterizing the pressure drop along the stream.

From (3.2) it follows that the terms of the series written out do not grow more rapidly than the terms of a geometrical progression. If the values of the coefficients $\mu_0$, and $\varkappa_0$ are of the same order, then the index of this geometrical progression will be given by the expression $A\varepsilon/(\gamma\mu_0)$. Estimating in this manner the initial terms of the series, we can make the corresponding assumptions about the accuracy of the approximate solution obtained.

Formulas (3.2) yield, in particular, a quantitative estimate for the effect of the compressibility of the medium at small pressure drops along the stream (when $A < 0$, the stream flows from left to right). Analysing the flow parameters we find it simplest if we retain, in formulas (3.2), terms up to and including $\varepsilon^3$; the pressure exhibits a small constant drop along the $Ox_1$ axis, and this can be regarded as the initial condition of the flow in question. The velocity of the medium is the same as that in Poiseuille flow. The pressure is constant across the flow, while the density and temperature both vary. Fig.4 shows the dependence of the quantity $\theta = 12\mu_0\varkappa_0T_2/A^2$ on $x_2$ when $h_0 = 1$. Lines *1-3* correspond to the values $\gamma = 1.1;\ 1.4;\ 7.$ When $\gamma \to +\infty$, the maximum value of $\theta$ does not tend to zero: $\max_{x_2}\theta \to 0.833$. The density and temperature behave differently in the downstream direction, i.e. when $x_1$ varies. In the case of the density, the "inhomogeneity amplitude" across the flow $(\max_{x_2}\rho - \min_{x_2}\rho)$ increases as $x_1$ increases, but does not change in the case of the temperature. Since the constant $\rho_{20}$ is chosen so that $v_{24} = 0$ when $x_2 = h_0$, it follows that $Q_2(h_0) = 0$. Therefore $\int \rho v_1 dx_2$, the flow of gas across the section $x_1 = \text{const}$ is given with an accuracy of up to and including $\varepsilon^3$, by the integral $\int v_{11}(x_2)dx_2$ and is analogous to the corresponding expression in the case of incompressible medium /1/ (the integration is carried out from 0 to $h_0$).

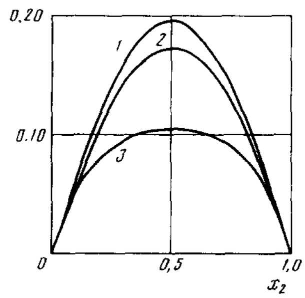


Fig.4

If we consider a tube of circular cross-section, then the representation (1.5) will have, in the axisymmetric case, the form

$$\begin{aligned} \rho = 1 + (\rho_{20} - T_2)\,\varepsilon^2 + (\rho_{30} - Ax_1\rho_2)\,\varepsilon^3 + \\ \{\rho_{40} + \gamma\mu_0A\,[v_{11}(r)\,\rho_2(r)]' - Ax_1\rho_3 - T_2\rho_2\}\,\varepsilon^4 + \\ \left[\rho_{50} + \gamma\mu_0\left(\frac{\partial v_{15}}{\partial r} + \frac{v_{15}}{r}\right) - Ax_1\rho_4 - T_2\rho_3\right]\varepsilon^5 + \varepsilon^6(\ldots) \end{aligned} \tag{3.3}$$

$$v_1 = \frac{A}{3\gamma\mu_0}(r^2 - r_0^2)\,\varepsilon + \left(\sum_{n=1}^{5} C_{2n}r^{2n}\right)\varepsilon^5 + \varepsilon^6(\ldots)$$

$$\begin{aligned} v_r = \left(\sum_{n=1}^{4} C_{2n-1}r^{2n-1}\right)\varepsilon^4 + \left[x_1\left(\sum_{n=1}^{4} L_{2n-1}r^{2n-1}\right) + \right. \\ \left.\sum_{n=1}^{2} M_{2n-1}r^{2n-1}\right]\varepsilon^5 + \varepsilon^6(\ldots) \end{aligned}$$

$$\begin{aligned} T_1 = 1 + Ax_1\varepsilon + \left\{\frac{A^2}{12\gamma\mu_0\varkappa_0}\left[\frac{(2-\gamma)}{4}r^2 - r_0^2\right]r^2 + T_{20}\right\}\varepsilon^2 + \\ \left(\sum_{n=0}^{4} L_{2n}r^{2n}\right)\varepsilon^4 + \left[x_1\left(\sum_{n=0}^{4} N_{2n}r^{2n}\right) + \sum_{n=0}^{2} M_{2n}r^{2n}\right]\varepsilon^5 + \varepsilon^6(\ldots) \end{aligned}$$

Here $r = \sqrt{x_2^2 + x_3^2}$; $v_r$ is the projection of the vector $\mathbf{V}(x_1, x_2, x_3)$ on the radius vector $\{0, x_2, x_3\}$; $C_n$, $L_n$, $M_n$, $N_n$ are const. Using (3.3) we can obtain formulas for the mean value of the velocity, flow of gas, etc., analogous to the corresponding formulas of the incompressible case /1/.

From (3.2) and (3.3) it follows that in the case of the motion of a compressible medium for small values of $A\varepsilon/(\gamma\mu_0)$, the Poiseuille laws describing the motion of a viscous fluid will hold approximately.

The problems of the stability of the flows (3.2) and (3.3) and of existence of their secondary solutions can be studied with help of the representation (1.5), taking the analogues of the Poiseuille flow constructed above as $\mathbf{U}_0$. Then we shall obtain for $\mathbf{U}_k$ linear systems with variable coefficients whose analysis is very time-consuming, merits a separate investigation, and is not dealt with here.

The stationary and non-stationary representations constructed describe, in an approximate manner, the solutions of specific initial boundary value problems. Using series (1.5) we can also obtain examples of separate flows of a viscous compressible gas. In particular, if we take as $\mathbf{U}_1$ (the solutions of the homogeneous system (2.1) for $k=1$)

$$\begin{aligned}
\rho_1 &= ({}^3/_2\gamma\mu_0 C \sin x_2 - B\cos x_2)\exp x_1 \\
v_{11} &= -C(\sin x_2 + x_2\cos x_2)\exp x_1 \\
v_{21} &= Cx_2\sin x_2 \exp x_1,\ T_1 = B\cos x_2 \exp x_1,\ B, C = \mathrm{const}
\end{aligned} \tag{3.4}$$

or a linear combination of the solutions (3.1) and (3.4), then we can write $\mathbf{U}_k$ $(k \geqslant 2)$ for the stationary case in explicit form.

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